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Solitons and symmetries

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Abstract. This article serves as an introduction to the special edition of Solitons and Symmetries in the Journal of Engineering Mathematics. Solitons, by their mathematical nature, are deeply connected to underlying symmetries of nonlinear equations. Described in this introduction is an historical and applications overview, with special emphasis directed at the technologically interesting field of soliton communications in fibre optics. A brief discussion of the papers contained in this special edition is included.

Key words: solitons, review, fibre optics, symmetries.

1. Background and overview

The last 30 years has seen an explosion of interest in a certain class of nonlinear equations. One of the primary reasons is the existence of special stable localized solutions called '*solitons*'. The history of solitons dates back to the work of Scott Russell [1, 2], Boussinesq [3, 4, 5], Korteweg and de Vries [6].

'Solitons' were first observed in 1834 by J. Scott Russell whilst riding on horseback beside the narrow Union Canal near Edinburgh, Scotland [1, 2]. Subsequently, Russell did extensive experiments in a laboratory scale wave tank in order to study this phenomenon more carefully. Russell observed a solitary wave – or in Russell's terminology, the 'great primary wave of translation,' which is a long, shallow, water wave of permanent form – and hence he deduced that they *exist*; this is his most significant result. This and subsequent investigations provoked much lively discussion and controversy as to whether the inviscid equations of water waves would possess such solitary wave solutions. From a mathematical point of view, the existence of these solitary waves was debated until the work of Boussinesq [3, 4, 5] – see Equation (30) p. 77 in [4] and Equations (283, 291) in [5] – and the extensive study by Korteweg and de Vries [6]. Korteweg and de Vries derived a nonlinear evolution equation governing long one-dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water, which after rescaling and nondimensionalization, may be written as

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The controversy was effectively resolved since this equation, now known as the Korteweg-de Vries (KdV) equation, has permanent wave solutions, including the *solitary wave solutions*

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2[\kappa(x - 4\kappa^2 t + \delta)],$$

where κ and δ are constants. Note that the velocity of this wave, $4\kappa^2$, is proportional to the amplitude, $2\kappa^2$ and so taller waves travel faster than shorter ones. We remark that Boussinesq also derived the nonlinear evolution equation

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx}, \tag{2}$$

which now bears his name and possesses the solitary wave solution

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2[\kappa(x - ct + \delta)], \qquad c^2 = 1 + 4\kappa^2.$$

The aforementioned Boussinesq Equation (2) has also been the study of considerable research (*cf.* [7, 8]). An asymptotic reduction of (2) yields the KdV Equation (1). In fact, there are many Boussinesq-like equations known. They have small parameters contained in them (before rescaling) and reduce to the KdV equation by asymptotic reduction.

Gardner and Morikawa [9] rediscovered the KdV Equation (1) in 1960 in the study of collision-free hydromagnetic waves. Subsequently the KdV equation has arisen in a number of other physical contexts including stratified internal waves in the ocean, ion-acoustic waves, plasma physics and lattice dynamics (*cf.* [8]). It should be remarked that water waves (and similarly for other applications) are only approximately described by the KdV Equation (1), in the limit of small amplitude and long waves.

The physical model which motivated the modern studies of the KdV equation is the Fermi-Pasta-Ulam (FPU) problem, which is a one-dimensional anharmonic lattice of identical masses coupled by nonlinear springs which Fermi, Pasta and Ulam numerically studied [10]. They felt that any smooth initial state would eventually relax into equilibrium due to the nonlinear springs. However a great surprise was encountered since the energy eventually recurred. That is, after flowing back and forth amongst the low order modes, the energy recollected into the lowest mode to within an accuracy of one or two percent, and from there on the process approximately repeated. In order to understand this phenomenon, Zabusky and Kruskal [11] studied a continuum model for the FPU problem. They found as continuum approximations first the Boussinesq Equation (2), and then the KdV Equation (1).

The remarkable properties of solitary wave solutions of the KdV Equation (1) were unknown until Zabusky and Kruskal [11] solved (1) numerically and made the critical discovery that the solitary wave solutions have the property that the interaction of two solitary wave solutions is elastic, *i.e.* they interact cleanly. When two solitary wave solutions of the KdV Equation (1) are initially well separated, with the larger to the left (assuming that they travel to the right), then the larger, faster one catches up with the smaller, slower one, and they overlap and interact nonlinearly. Zabusky and Kruskal discovered that after the interaction, the waves separate, with the larger one on the right, having regained their initial amplitudes and velocity. The only effect of the interaction is a phase shift, so that the peaks of the waves are at different positions from where they would have been without the interaction.

At the centre of the observations of Zabusky and Kruskal [11] is the discovery that these nonlinear waves can interact elastically and continue afterwards almost as if there had been no interaction at all. Because of the analogy with particles, Zabusky and Kruskal called these solitary waves *solitons*. Their remarkable numerical discovery demanded an analytical explanation and detailed mathematical study of the KdV equation. However the KdV Equation (1) is nonlinear and at that time no general method of solution for nonlinear equations was known.

We remark that there is an alternative shallow water wave model

$$u_t + u_x + 6uu_x - u_{xxt} = 0, \qquad (3)$$

proposed by Peregrine [12] and Benjamin, Bona and Mahoney [13], which also has a solitary wave solution

$$u(x,t) = \frac{2\kappa^2}{1-4\kappa^2}\,\mathrm{sech}^2[\kappa(x-ct+\delta)], \qquad c = \frac{1}{1-4\kappa^2}.$$

Numerical studies show that the interaction of the solitary waves of Equation (3) is inelastic [14]. Equation (3) also differs from the KdV equation in that the KdV equation is asymptotically 'unique' in the sense that all small parameters are removed via the analysis leading to the equation. An asymptotic reduction of Equation (3) yields the KdV equation.

Subsequently, Gardner, Greene, Kruskal and Miura [15] developed the *inverse scattering method* to solve the initial value problem for the KdV Equation (1) and via this method explained the occurrence of soliton solutions. Gardner, Greene, Kruskal and Miura associated the solution of the KdV equation with the time independent Schrödinger equation

$$\phi_{xx} + u(x,t)\phi = \lambda\phi,$$

(here t is a parameter) and showed, using the ideas from the theory of direct and inverse scattering, that the Cauchy problem for the KdV Equation (for initial data on the line which decays sufficiently rapidly), could be solved in terms of the solution of a *linear* integral equation.

Subsequently this has evolved into a new method in mathematical physics which can be viewed as a nonlinear extension of Fourier transforms. The method is now often referred to as the *Inverse Scattering Transform* (IST), and numerous physically significant nonlinear equations have been solved by generalizations of this technique (*cf.* [7, 8]) which include the following:

- partial differential equations in one spatial and one temporal dimensions,
- differential-difference equations,
- ordinary differential equations,
- ordinary difference equations,
- singular integro-differential equations,
- partial differential equations in two spatial and one temporal dimensions.

An evolution equation, *e.g.* a partial differential equation, that is solvable by some form of the inverse scattering method is often referred to as being *completely integrable*, this terminology arising from the interpretation of the KdV equation as a completely integrable Hamiltonian system [16]. Such integrable evolution equations have been shown to usually possess several remarkable properties including:

- the 'elastic' interaction of solitary waves, *i.e.* multi-soliton solutions,
- Bäcklund transformations,
- an infinite number of independent conservation laws,
- a complete set of action-angle variables,
- an underlying Hamiltonian formulation,
- a Lax representation,

- a bilinear representation and soliton solutions via a finite series of exponential solutions, and
- an associated linear eigenvalue problem whose eigenvalues are constants of the motion.

(for definitions and details see *e.g.* [7, 8]). However, the precise interrelationships between all these (and other) properties has yet to be rigorously formulated. A fundamental open question is, *what is it that really characterises completely integrable equations?*

One of the important aspects of 'soliton theory' is the connection between the integrals of motion which are related to symmetries of the associated nonlinear equation which in turn give rise to the solitons. There are many mathematical results in the theory of solitons which interrelate Mathematics and Physics and in our opinion this is the beauty of the subject! During the past thirty years or so, soliton equations have been related with and have attracted considerable interest from many branches of mathematics including differential geometry, Hamiltonian mechanics, group theory, partial differential equations, and numerical analysis. There are numerous important applications of soliton theory which have spurred the interest of physicists and engineers.

2. Applications perspective

As well as having a rich mathematical structure, solitary waves (including solitons) have been obtained as solutions to nonlinear equations modelling a variety of circumstances. For example solitons have been derived in the following physical applications:

- Water waves in channels, shallow water and the ocean (Scott Russell was a naval architect who designed the Great Eastern, a sailing ship; the design of which was partly based on his knowledge of the dynamics of solitons).
- Lattice dynamics (*e.g.* the FPU model); waves in rods and strings
- Electrical transmission lines.
- General relativity.
- Josephson junctions and superconductors.
- Liquid crystals.
- Optical fibres and telecommunications; nonlinear optics.
- Plasma physics.
- Protein dynamics and DNA.
- Quantum field theory.
- Stratified fluids.
- Rossby waves.
- Statistical mechanics.

3. Nonlinear fibre optics

One of the most important applications of soliton theory is in the study of nonlinear fibre optics which we shall discuss in this section.

Following the discovery of the inverse scattering method for the KdV equation, in 1971 Zakharov and Shabat [17] developed the method of solution for the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t + u_{xx} + \sigma |u|^2 u = 0, \qquad \sigma = \pm 1, \tag{4}$$

which had been considered earlier by researchers in water waves (*cf.* Benney and Newell [18], Benney and Roskes [19], Zakharov [20]).

In 1973, Hasegawa and Tappert [21, 22] discussed the relevance of the NLS Equation (4) in optical fibres and their associated solitary wave solutions. They did computer simulations to demonstrate the stability of these solitary waves and discussed how the NLS Equation (4) described the instabilities of wave packets in fibre optics. Hasegawa and Tappert showed that optical fibres could sustain envelope solitons – both bright and dark solitons; *i.e.* bright with anomalous (positive) dispersion and dark with normal (negative) dispersion. These solitons propagate in the longitudinal dimension having a single mode guided in the direction perpendicular to the propagation direction. Whilst Hasegawa and Tappert were working on the subject, Zakharov and Shabat [17] published their paper on the solution of the initial value problem of the NLS Equation (4). Although the Zakharov and Shabat paper was published in 1972, Hasegawa and Tappert were unaware of it until they had completed their studies. The results of Zakharov and Shabat [17] actually confirmed the conjecture by Hasegawa and Tappert [21, 22] of stable bright nonlinear pulse transmission of envelop light waves in optical fibres.

In the early 1980's, Mollenauer, Stolen and Gordon [23, 24] (see also [25] showed that solitons could be produced in laboratory experiments. They observed the narrowing of the light wave pulse as the input power was increased, thereby verifying the soliton phenomenon in the fibre. Optical solitons in fibres form as the nonlinearity balances the *dispersive* spreading of a guided wavepacket.

However, there was a serious problem, namely that the amplitude decreased by a factor of 10 over 100 km. Several proposals were made to overcome this problem. In 1987 suitable amplifiers, *i.e.* Erbium–Doped Fibre Amplifiers (EDFA) [26, 27], which counteracted the dissipation, were developed. Thus the idea of an all optical transmission system became realistic. It was found that successful long distance transmission with repeated amplifications in a non-adiabatic regime required enhancement of the initial amplitude and that for this application, the amplifier distance is much smaller than the dispersion distance (for a detailed discussion, see, for example, [28, Chapter 7]). Based upon these scales, allowing for damping and amplification, Kodama and Hasegawa [29] developed a theory in which the NLS equation with its ideal (bright) soliton solution, is the leading order approximation.

Unfortunately these solitons have some serious technological defects, one of them due to amplifier noise. Effective long distance soliton transmission system utilizing only optical amplifiers was shown to be problematic in the study of Gordon and Haus [30]. They demonstrated that amplifier noise would induce soliton velocity variations which in turn creates arrival time jitter. This noise/jitter effect is known as the '*Gordon-Haus effect*' or '*Gordon-Haus jitter*'. In the 1990's, several techniques were developed to overcome this problem. Frequency filters [31, 32] and sliding frequency filters which continuously shift the central frequency of the filters [33] were proposed as mechanisms to alleviate Gordon-Haus jitter.

The technological challenge of transmitting large amounts of information via solitons is further complicated by the need to send many solitons at the same time (namely, multi-soliton transmission) in the same fibre, but in different frequency channels (*i.e.* the solitons have different carrier frequencies). This technique is referred to as wavelength division multiplexing (WDM). Invariably this means that solitons undergo many collisions as they travel long distances down a fibre and therefore one has yet another important technological problem to deal with: the effects of collisions of solitons. In the references [34, 35] it has been shown that WDM soliton collisions produce resonant growth of waves in neighbouring frequency channels via what is usually referred to as four wave mixing (FWM). These FWM waves potentially can destroy one's ability to measure a soliton, or a 'bit,' in a realistic transmission system.

An important related issue is the timing jitter caused by WDM soliton collisions [38]. In recent work (*cf.* [36, 37]) it is has been shown that the timing jitter in WDM systems is greatly reduced by employing filtering and a method referred to as dispersion management. Dispersion management is a technique in which fibres with different dispersion characteristics are merged together. The type of dispersion management described in the above papers is called dispersion management following loss profiles. The aim here is to approximate the dissipation loss with fibres of suitable dispersion characteristics. In these papers it is shown that this type of dispersion management is useful in being able to reduce timing jitter due to soliton collisions.

Another type of dispersion management technique which has received considerable attention during the past few years is what is often referred to as strong periodic dispersion management. This method employs merged fibres with large changes in dispersion characteristics and with alternating signs; *i.e.* concatenated fibres consisting of both positive (anomalous) and negative (normal) subportions. Researchers have shown that such configurations can substantially reduce Gordon-Haus jitter as well as collision induced timing jitter (*cf.* [39–50]). It also appears that this type of dispersion management may be easier to implement in practical transmission lines [51].

In these cases the slowly varying wave theory which originally had led to the NLS equation as being a fundamental equation (*cf.* [52, 53]) has to be re-examined. In a recent paper [54] it was shown, via the method of multiple scales, that a nonlocal NLS equation governs the leading order solution and that this equation has solitary travelling wave solutions with oscillating, decaying tails. The central part of the solitary wave solution has a Gaussian-like core.

Currently there is extensive research and interest in strong periodic dispersion management and it is possible that systems of this type will be used for new long distance transmission lines.

Although the original motivation to create solitons in optical fibres was for use in communications, the nonlinear phenomena which arise in optical fibres have also been used to develop new laser sources, measuring devices, sensors and switches. Several of these applications yield mathematically interesting systems of differential equations. Nonlinear optics is an example of the beauty and practical importance of nonlinear phenomena and the interrelation between Mathematics, Physics and Engineering.

Much of the detailed mathematical analysis of soliton communication systems in which the NLS equation plays an important role can be found in the book by Hasegawa and Kodama [28]. Other books which discuss analytical techniques in nonlinear optics and optical solitons include [55, 56, 57].

4. The special Edition

Our point of view is to encourage the interaction of Mathematics, Physics, Engineering and whenever possible, technology. We believe the varying views and methods from different fields has proven to be particularly fruitful in the study of soliton theory. As mentioned above, solitons and symmetries are closely aligned. Besides the fact that the integrals of motion of soliton equations are related to (so called) higher order symmetries, we also note that the well known Painlevé equations are related to scaling (Lie) symmetries (*cf.* [7, 8]) of soliton equations. The Painlevé equations are important in the analysis of the long time asymptotic structure of many of the well known nonlinear evolution equations (*e.g.* the KdV, modified KdV, sine-Gordon equations) and arise in many physically interesting contexts.

In this special edition of the Journal of Engineering Mathematics we have collected articles from diverse, but nevertheless related, fields. These papers discuss fundamental issues such as symmetries and solutions associated with nonlinear equations, as well as the underlying analysis and applications of solitons in fluid mechanics and optics. Researchers in applied mathematics have found that perturbation methods are often critical in developing a suitable mathematical understanding of the phenomena associated with physical problems. The paper of T-S. Yang and W. Kath and the paper by C. Menyuk employ multiple scale perturbation methods which yield relevant nonlinear equations in fibre optics, *e.g.* the NLS equation in the presence of dispersion variation and the coupled NLS equations when polarization effects are taken into account. D. Calvo and T. Aklylas use exponential asymptotics to describe radiation effects associated with soliton solutions of certain equations. The paper of A. Ankiewicz, N. Achmediev and P. Winternitz discuss solitary wave solutions associated with certain physically interesting nonlinear (Ginzburg-Landau type) equations and their analytical structure in the complex plane. R. Grimshaw and S. Pudjaprasetya investigate a problem where solitary waves propagate on a variable background. This problem can arise in water waves evolving over uneven topography. They analyse a variable-coefficient Korteweg-de Vries Equation, its slowly varying solitary wave solution and the underlying Hamiltonian structure.

P. Broadbridge discusses exact solutions of a forced Burgers' Equation which models unsaturated flow in the presence of plant roots. Applying the standard Cole-Hopf transformation yields a linear partial differential equation with non-constant coefficients which is solved using separation of variables in terms of a stationary Schrödinger equation. S. C. Davies, J. R. King and J. A. D. Wattis describe solutions to the Smoluchowski coagulation equations with power law kernels in both constant mass and constant monomer cases. Exact solutions are obtained in special cases by a generating function approach and numerical results are also given, which confirm the asymptotic analysis. Yi. A. Li, B. T. N. Gunney and P. J. Olver study the properties of solitary wave solutions of a particular fifth order evolution equation that models water waves with surface tension. Existence and nonexistence results are surveyed and strengthened. An accurate numerical code is devised and used to show the small dispersive effects of solitary wave collisions. J. J. C. Nimmo, C. Rogers and W. K. Schief apply a truncation of Bergman-type series to a canonical hyperbolic equation which yields the integrable B_n Toda lattice. In particular, the recurrence relations for the truncated Bergman series are implied by the linear representation for the Toda system. To conclude, the action of the Moutard transformation on Bergman series is investigated. D. F. Parker and E. N. Troy study solutions of coupled nonlinear partial differential equations describing optical cascading. In particular they construct periodic analogues of some classes of solitary wave solutions which are expressed in terms of Jacobian elliptic functions.

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